# Introduction to Verified Numerical Computation (Verified Computing)

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# System of linear equations

- Basic and important problems in linear algebra and numerical analysis
  - There are enormous number of problems of finding multi unknown variables.
  - The simplest and most typical model is system of linear equations.
  - Solving a system of linear equations is required to solve a system of nonlinear equations in most cases.

Example)

$$\begin{cases} 2x + 4y + 6z = 6 \\ 3x + 8y + 7z = 15 \\ 5x + 7y + 21z = 24 \end{cases} \quad \text{can be written as} \quad Ax = b$$

where

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 8 & 7 \\ 5 & 7 & 21 \end{bmatrix}, \qquad b = \begin{bmatrix} 6 \\ 15 \\ 24 \end{bmatrix}.$$

# Numerical methods for solving linear equations

Numerical method for solving a system of linear equations are classified into “Direct methods” and “Iterative methods”.

## Direct methods

- Gaussian elimination method or LU-factorization method
- Cholesky method (for a symmetric matrix)

- Advantage
  - Stable (compared with iterative methods)
  - Wide application range
  - Applicable for both dense matrices and sparse matrices

- Disadvantage
  - A lot of memory
  - Long computation time with O($N^3$) order

⇒ Mainly used for dense problems

# Numerical methods for solving linear equations

## Iterative methods

Iterative methods are further classified into "Stationary" and "Nonstationary".

- Stationary iterative methods
  - Jacobi method
  - Gauss-Seidel method
- Nonstationary iterative methods
  - Several types of Krylov subspace methods
  - For example, CG method

Note:
CG method: Conjugate Gadient method

- Advantage
  - Small memory usage
  - Short computation time
- Disadvantage
  - Limited application range (mainly used for sparse matrices)

# Exercise using VCP Library

- Run the test codes for matrices.
- Try the Gaussian elimination method with interval arithmetics using lss packaged in VCP Library.
- Observe the increase of the width of intervals.

test_matrix.cc
test_interval_matrix.cc
approx_gauss.cc
interval_gauss.cc

# Fundamental theorem for $Ax = b$

- $I$: $n \times n$ identity matrix
- $A$: $n \times n$ real matrix
- $b$: $n \times 1$ real vector
- $x^*$: the solution of $Ax = b$
- $\tilde{x}$: approximate solution of $x^*$
- $R$: approximation of $A^{-1}$

**Theorem 5.3**

Suppose that

$$\|RA - I\| < 1.$$

Then, we have

$$\|x^* - \tilde{x}\| \leq \frac{\|R(b - A\tilde{x})\|}{1 - \|RA - I\|}.$$

verify_lineareq.cc

# Yamamoto's theorem for $Ax = b$

- For matrix (vector) $M = [m_{i,j}]$, we denote $|M| = [|m_{i,j}|]$.
- $e = [1,1,\cdots,1]^T$.

**Theorem 5.6**

Suppose that

$$\|RA - I\| < 1.$$

Then, we have

$$|x^* - \tilde{x}| \leq |R(b - A\tilde{x})| + \frac{\|R(b - A\tilde{x})\|_\infty}{1 - \|RA - I\|_\infty} |RA - I| e$$

T. Yamamoto, Error bounds for approximate solutions of systems of equations, *Japan Journal of Applied Mathematics,* Vol 1, 157-171 (1984).

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