Introduction to Verified Numerical Computation (Verified Computing)

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Interval arithmetic

History:

- The origin of interval arithmetic is the seminal master thesis by **Teruo Sunaga (須永 照夫)**, handwritten in Japanese, submitted on February 29, 1956.
- R. E. Moore wrote a book concerning interval arithmetic in 1966.
- S. M. Rump and S. Oishi extended it to fast arithmetics for vectors and matrices around 2000.

Interval arithmetic

$$\pi \in [3.14, 3.15]$$
$$\sqrt{2} \in [1.41, 1.42]$$

Interval arithmetic →

Round up

$$\pi + \sqrt{2} \in [4.55, 4.57]$$

Round down

Interval arithmetic

Example of implementation:

- Let $\mathbb{F}$ be the set of floating point numbers.
- For $x = [\underline{x}, \overline{x}]$ and $y = [\underline{y}, \overline{y}]$ ($\underline{x}, \overline{x}, \underline{y}, \overline{y} \in \mathbb{F}$), we calculate

$$x + y = \left[\triangledown\left(\underline{x} + \overline{x}\right),\ \triangle\left(\underline{y} + \overline{y}\right)\right],$$

where $\triangledown$($\triangle$) stands for round down (up).

- See Definition 3.2 in Sekine’s note for the other arithmetics.

Try interval arithmetic!

interval_test.cc

Exercise

Calculate the range of the function

$$f(x) = x + \sin(\log(x) + x^2)$$

for $1 \leq x \leq 4$ using interval arithmetics.

Is interval arithmetic always the best?

NO

- Repetition of the interval operations increases the width of intervals exponentially.
- A famous example is the Gaussian elimination method with interval arithmetics (will be proposed later).

> This does not imply that interval arithmetic is useless, but it does place severe restrictions on the way it can be applied. In general it is best in algebraic computations to leave the use of interval arithmetic as late as possible so that it effectively becomes an a posteriori weapon [21].

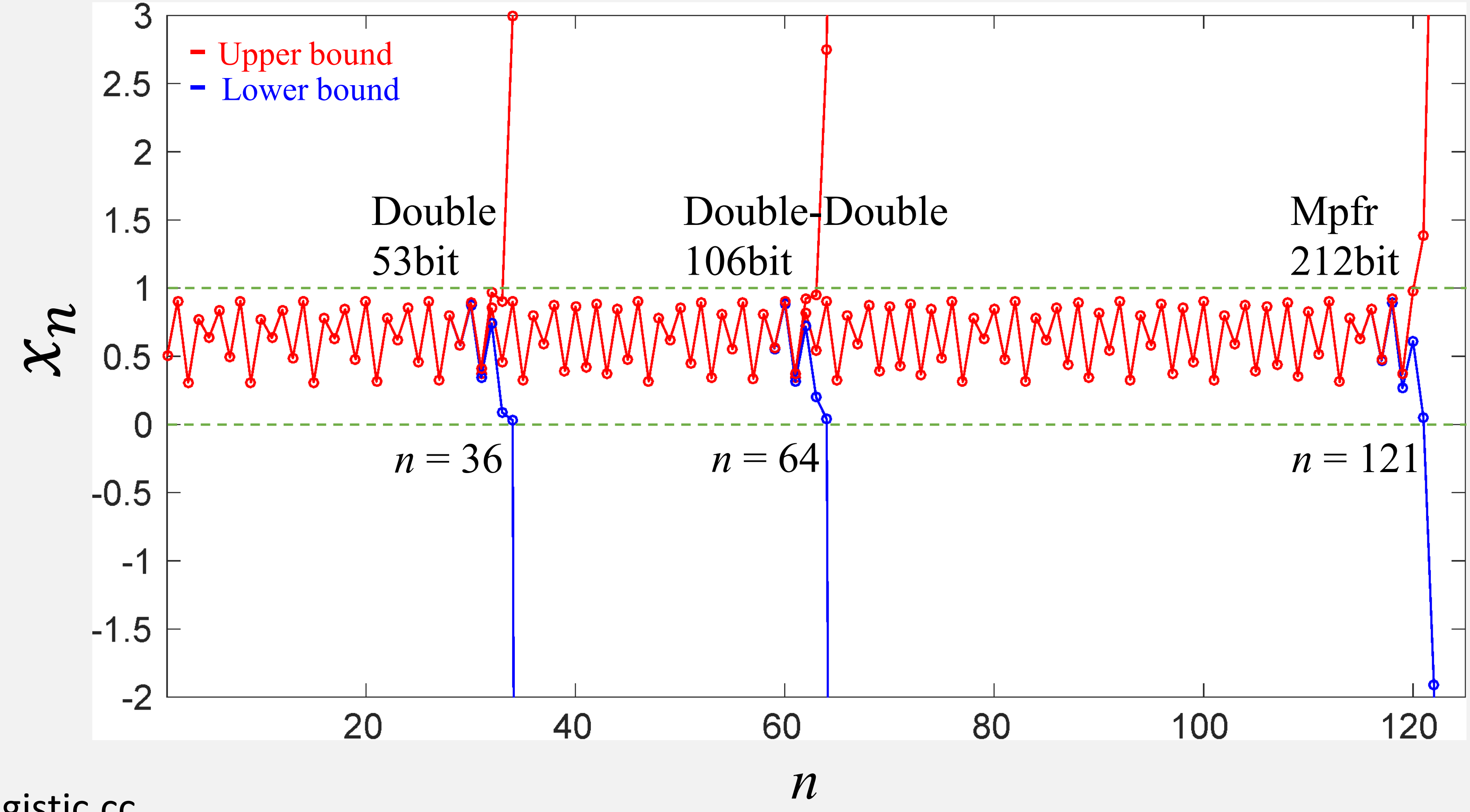
James H. Wilkinson, Modern error analysis, *SIAM review,* 13, 4, 548-568 (1971).

Logistic map

$$x_{n+1} = ax_n(1 - x_n)$$

$$\begin{cases} a_c < a \leq 4, \ a_c \approx 3.57 \\ 0 < x_0 < 1 \end{cases}$$ → causes chaos.

$a = 3.625, \ x_0 = 0.5$



logistic.cc