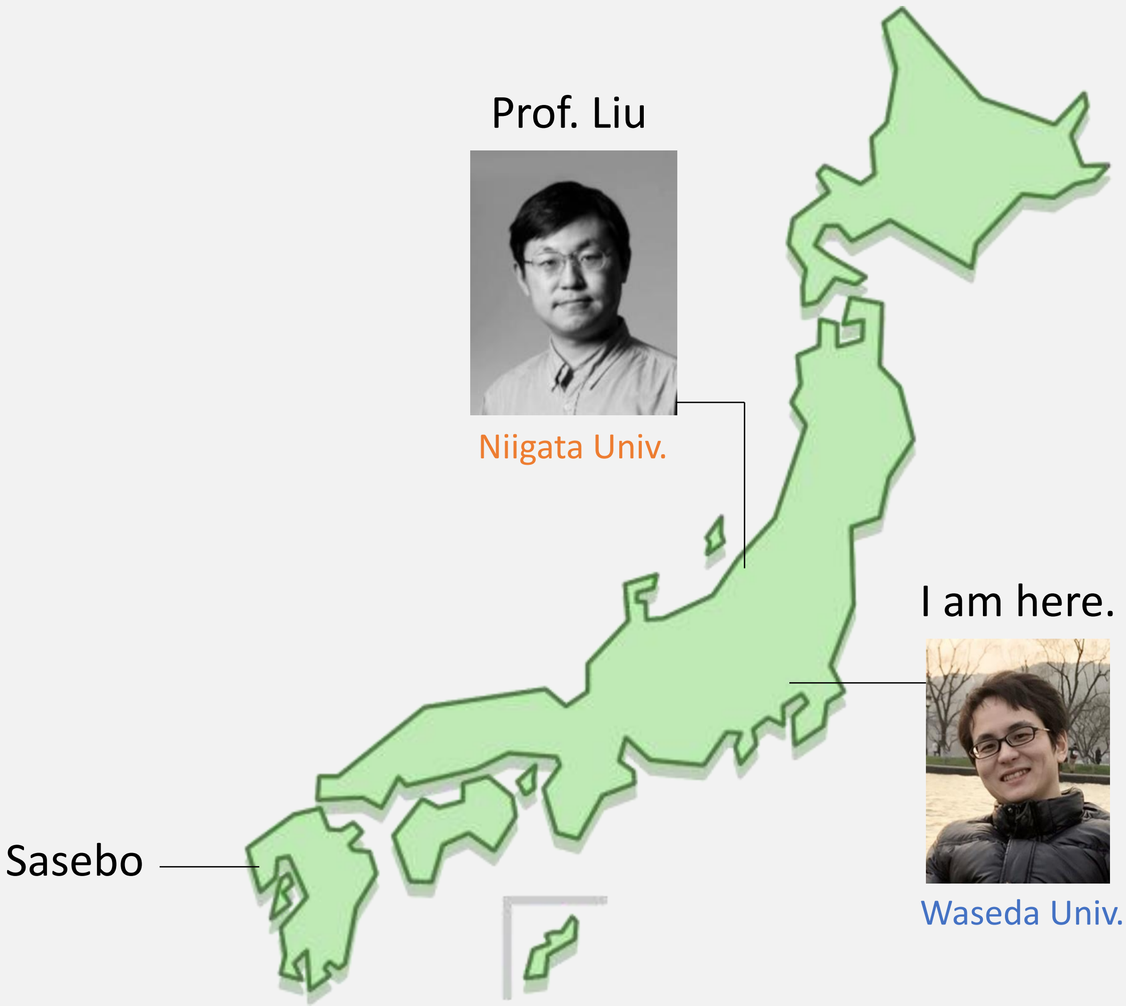
Introduction to Verified Numerical Computation (Verified Computing)

TANAKA, Kazuaki
田中 一成
tanaka@ims.sci.waseda.ac.jp
Institute for Mathematical Science, Waseda University
LAB: Building No.60, Room 307B

About me

- Name: TANAKA, Kazuaki / 田中 一成
- Birthplace: Sasebo, Nagasaki / 佐世保市, 長崎県
- Education:
  - Apr. 2008 – Mar. 2012: Department of Applied Mathematics, Waseda University.
  - Apr. 2012 – Mar. 2014: Department of Pure and Applied Mathematics, Waseda University (Master).
  - Apr. 2014 – Mar. 2017: Department of Pure and Applied Mathematics, Waseda University (Doctor).
- Career
  - Apr. 2017 – Mar. 2018: Assistant Professor, Department of Applied Mathematics, Waseda University.
  - Apr. 2017 – Mar. 2018: Assistant Professor, **Institute for Mathematical Science**, Waseda University.

Our location



Support

- Taisei Asai (Waseda University)

  Doctor Course, Fundamental Science and Engineering

  Research Topic: Verified Numerical Computations for PDEs

- Shuusuke Tada (Waseda University)

  Under graduated, Fundamental Science and Engineering

  Research Topic: Verified Numerical Computations for Mean Curvature Flow

Keywords

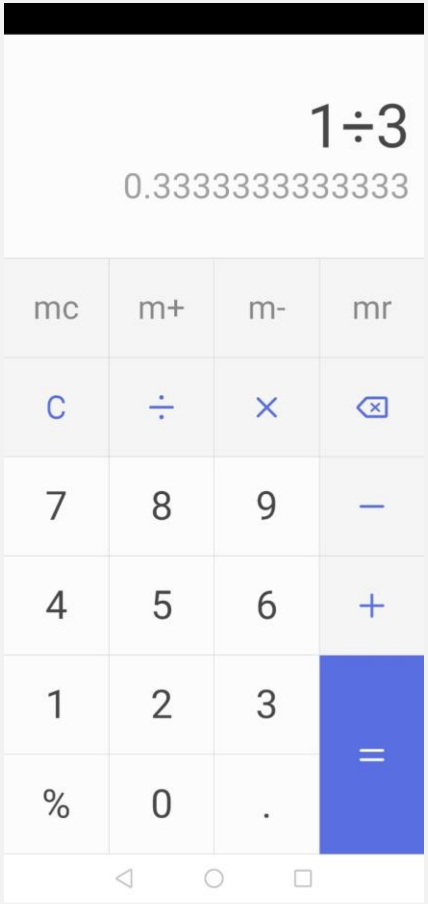
- **Verified numerical computation** / 精度保証付き数値計算
- Numerical verification method / 数値的検証法
- Computer-assisted proofs (analysis) / 計算機援用証明（解析）
- **Verified computing**
- Reliable computing
- Rigorous numerics
- Validated numerics
- Self-validating numerical methods
- Self-validated numerical methods

Verified Numerical Computation

- Numerical methods strictly estimating errors therein, for example, **rounding errors**, truncation errors, and discretization errors.
- Verified numerical computation adds mathematical reliability to computation results.
- **Interval arithmetics** plays an important role for implementing verified numerical computations.

Example of Errors

Calculator in your phone



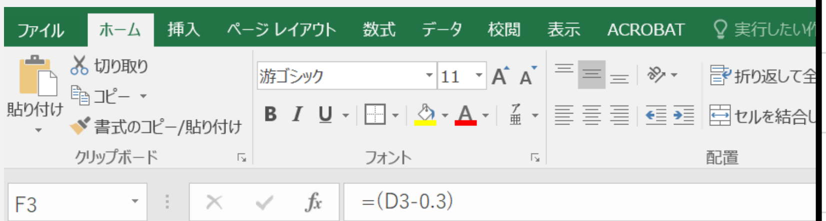
The exact result is

0.33333333333333333333.........

which has infinite digits.
We have to round it off to finite digits.

Example of Errors

Excell



	A	B	C	D	E	F
1						
2		a	b	a-b		
3		10.3	10	0.30000000000000100		
4		9.3	9	0.30000000000000100	0.00000000000000072164	0.00000000000000072164
5		8.3	8	0.30000000000000100	0.00000000000000072164	0.00000000000000072164
6		7.3	7	0.30000000000000000	0.00000000000000000000	-0.00000000000000016653
7		6.3	6	0.30000000000000000	0.00000000000000000000	-0.00000000000000016653
8		5.3	5	0.30000000000000000	0.00000000000000000000	-0.00000000000000016653

a	b	a-b
10.3	10	0.30000000000000100
9.3	9	0.30000000000000100
8.3	8	0.30000000000000100
7.3	7	0.30000000000000000
6.3	6	0.30000000000000000
5.3	5	0.30000000000000000

*** Office 2016(W), Office365(W), Office for Mac**

Confirm with C++

welcome.cc
example1.cc

Error is serious?

Significant discrepancies (between the computed and the true result) are very rare, too rare to worry about all the time, yet not rare enough to ignore.

W.M.Kahan

很多问题的计算结果和真实结果之间的重大误差很少发生，大部分情形不必担心，但是这种误差并不是可以完全忽略。

数値計算結果と正しい結果の間に著しい食い違いがあることは極めて稀である。極めて稀であるため、常に心配する必要はない。ただ、無視できるほどに稀というわけでもない。

* 日本語訳は以下より引用
中尾充宏, 渡部善隆: 実例で学ぶ精度保証付き数値計算: 理論と実装. サイエンス社, 2011.

From Wikipedia

Q. What is the value of x?

```
x = 100
for i = 1 to 60
    x = sqrt(x)
end
for i = 1 to 60
    x = x^2
end
```

a. x=1000 ← Result from computers

b. x=1

c. x=10 ← Analytical Result

d. x=100

e. x=0

Computers return a wrong result.

example2.cc
example2_highprecision.cc

Rump's example

Rump found the following example

$$f(a,b) = (333.75 - a^2)b^6 + a^2(11a^2b^6 - 121b^4 - 2) + 5.5b^8 + \frac{a}{2b}.$$

Try to calculate this function for $a = 77617$ and $b = 33096$.

The exact solution is -0.827396…..

S.M. Rump. Algorithms for Verified Inclusions - Theory and Practice. In R.E. Moore, editor, *Reliability in Computing*, volume 19 of *Perspectives in Computing*, pages 109–126. Academic Press, 1988.

example3.cc

Spurious solution of Emden’s equation

- Breuer-Plum-McKenna observed a spurious solution of Emden’s equation $-\Delta u = u^2$.
- The existence of asymmetric solutions has been denied by Gidas-Ni-Nirenberg’s theory.

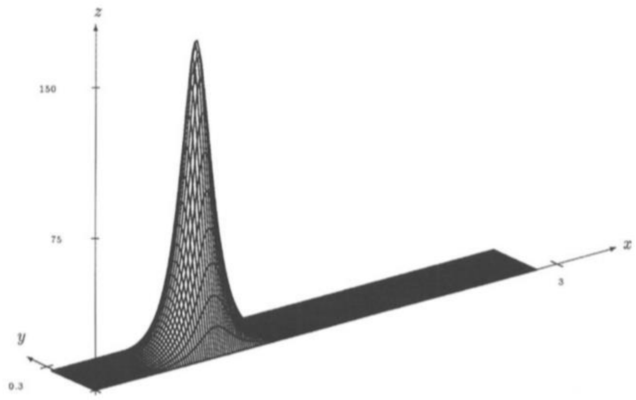


Figure 6.3. “Spurious” approximate solution of (22) for $\zeta = 2.9$

$$\Delta u + u^2 = 0$$

B. Breuer, M. Plum, and P. McKenna, “Inclusions and existence proofs for solutions of a nonlinear boundary value problem by spectral numerical methods,” in Topics in Numerical Analysis , pp. 61–77, Springer, 2001

Interval arithmetic

$$\pi \in [3.14, 3.15]$$
$$\sqrt{2} \in [1.41, 1.42]$$

Interval arithmetic →

$$\pi + \sqrt{2} \in [4.55, 4.57]$$

Round down (4.55) / Round up (4.57)

Q. What is the value of x? - Interval version -

Double (53bit) → x = [1, 2.2844e+222]

Double-Double (106bit) → x = [99.999, 100.01]

example2_interval.cc

Today's Goal

- To understand the concept of verified numerical computation (verified computing).
- To obtain the fundamental knowledge of floating-point numbers and understand how to use simple interval arithmetics.
- Apply such techniques to system of linear equations.